NORTH-HOLLAND

On $(A, B)^t$-Invariant Subspaces Having Extendible Brunovsky Bases

A. Compta* and J. Ferrer†
Departament de Matemàtica Aplicada I
E.T.S. Enginyers Industrials de Barcelona
Universitat Politècnica de Catalunya
Diagonal 647
08028 Barcelona, Spain

Submitted by Leiba Rodman

ABSTRACT

We consider $(A, B)^t$-invariant subspaces having a Brunovsky basis which can be extended to a Brunovsky basis of the whole space. We obtain a geometrical characterization of this class of $(A, B)^t$-invariant subspaces, and a complete family of numerical invariants to classify them. © Elsevier Science Inc., 1997

1. INTRODUCTION

For A a square matrix, Gohberg et al. [6] introduce an "interesting class" of A-invariant subspaces, which they call "marked": an A-invariant subspace is marked if and only if it has a Jordan basis which can be extended to a Jordan basis of the space. The A-marked subspaces have been studied, for example, in [2] and [5].

Moreover, for (A, B) a pair of matrices, [6], [3] and other works define and study the (A, B)-invariant subspaces and the $(A, B)^t$-invariant subspaces.

* E-mail: compta@ma1.upc.es.
† We are very grateful to F. Puerta and X. Puerta for their valuable comments and suggestions. We also thank the referees for their careful review.

LINEAR ALGEBRA AND ITS APPLICATIONS 255:185–201 (1997)

0024-3795/97/$17.00
PII S0024-3795(96)00001-8

The question naturally arises of the existence of Brunovsky bases of those subspaces which can be extended to Brunovsky bases of the space. (A basis is called a *Brunovsky basis* if relative to it the matrix has the Brunovsky reduced form.) That is to say, we wish to consider (A, B)-marked and $(A, B)^t$-marked subspaces.

This paper is a geometrical approach to the study of $(A, B)^t$-marked subspaces. In fact, following the techniques in [4], we consider linear maps $f: Y \to X$ defined only on a subspace of Y of X. Then we say that a subspace $W \subset Y$ is f-invariant if $f(W) \cap Y \subset W$, and among these subspaces we call f-marked those having a extendible Brunovsky basis.

The first goal is a geometrical characterization of the f-marked subspaces. In [5], such a characterization has been obtained for f an endomorphism. Here, Theorem 5.1 generalizes this result by means of the additional condition

$$\left[W \cap f^{j-i}(f^{-j+1}(Y))\right] + Y_i = (W + Y_i) \cap \left[f^{j-i}(f^{-j+1}(Y)) + Y_i\right]$$

for all $1 \leqslant i \leqslant j$.

The proof of this theorem includes the construction of a Brunovsky basis of W and its extension to a Brunowsky basis of $f: Y \to X$, provided that the above conditions are verified. This explicit construction suggests a condition to guarantee the existence of a bijection between these kind of bases for two different f-marked subspaces. Thus, the second main result is Theorem 6.1, which classifies the $(A, B)^t$-marked subspaces by means of a complete family of numerical invariants.

Section 2 contains the notation which will be used in the paper.

In Sections 3 and 4 we present the definition and some generalities of $(A,B)^t$-invariant and $(A, B)^t$-marked subspaces respectively. In particular, in Section 3.5 we characterize the Brunovsky invariants of an f-invariant subspace. This result has been proved in [3] by means of matricial techniques.

Section 5 is devoted to proving Theorem 5.1.

In Section 6 we define a natural equivalence between $(A, B)^t$-invariant subspaces, and we prove the classification Theorem 6.1. As an application, in Section 6.7 we compute the number of nonequivalent f-marked subspaces for a fixed linear map $f: Y \to X$.

2. PRELIMINARIES

2.1

X will be an $(n + m)$-dimensional vector space over the complex numbers $\mathbb{C}$.

If B is a subset of X, the symbol $[B]$ will denote the subspace spanned by the vectors of B.

If $X \supset E_1 \supset E_2 \supset \cdots$ is a chain of subspaces of X, we say that a basis B of X is *adapted* to it if $B \cap E_i$ is a basis of E_i for all $i = 1, 2, \ldots$.

2.2

$Y \subset X$ will be an n-dimensional subspace, and $f: Y \to X$ a linear map defined on it. In any basis of X adapted to $Y \subset X$, the matrix f is a pair

$$\begin{pmatrix} A \\ B \end{pmatrix} \qquad \text{where} \quad A \in M_n(\mathbb{C}) \text{ and } B \in M_{m,n}(\mathbb{C}).$$

As usual, we will say that two linear maps defined on a subspace, $f: Y \to X$ and $f': Y' \to X'$, are equivalent (or *block-similar*) if there is an isomorphism $\varphi: X \to X'$ such that $\varphi(Y) = Y'$, $\varphi \circ f = f' \circ \hat{\varphi}$, where $\hat{\varphi}$ is the restriction of φ to Y. It amounts to saying that their respective matrices

$$\begin{pmatrix} A \\ B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A' \\ B' \end{pmatrix}$$

are block-similar.

2.3

Following [4], we will consider the stationary chain of subspaces and linear maps defined as follows:

$$Y_0 \equiv Y, \qquad Y_i = f^{-1}(Y_{i-1}), \quad i > 0,$$

$$Y_{k+1} = Y_k \subset Y_{k-1} \subset \cdots \subset Y_1 \subset Y \subset X,$$

$$f_i : Y_i \to Y_{i-1}, \qquad i > 0,$$

where f_i is the corresponding restriction of f. We will denote f_i simply by f if no confusion is possible.

2.4

We recall that a complete family of invariants of the equivalence in Section 2.2 is formed by:

(1) The Kronecker indices $(k_1, \dots, k_r)$, which can be computed as the conjugate partition of $(r_1, \dots, r_k)$, where

$$r_i = \dim Y_{i-1} - \dim Y_i, \qquad i = 1, \dots, k.$$

(2) The similarity invariants of the endomorphism

$$f_{k+1} : Y_{k+1} = Y_k \to Y_k$$

or equivalently its invariant factors $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_t$.

In particular, $f : Y \to X$ is observable if and only if $Y_k = \{0\}$.

2.5

Also, we recall that a Brunovsky basis of $f : Y \to X$ is formed by a Jordan basis of $f_{k+1} : Y_k \to Y_k$ joint with r so-called *Brunovsky chains*

$$x_j, f(x_j), \dots, f^{k_j}(x_j) \notin Y, \qquad 1 \leqslant j \leqslant r,$$

such that $f^{k_1}(x_1), \dots, f^{k_r}(x_r)$ are linearly independent. They can be obtained by taking a Jordan basis of Y_k and extending it successively in the chain of subspaces $\cdots \subset Y_i \subset \cdots$ by taking images and extending step by step—that is to say, by means of supplementary subspaces $\bar{Y}_i$ $(i = k, \dots, 1, 0)$ such that

$$\begin{aligned}
&Y_{k-1} = Y_k \oplus \bar{Y}_k, \\
&Y_{k-2} = Y_{k-1} \oplus \bar{Y}_{k-1}, \qquad \bar{Y}_{k-1} \supset f(\bar{Y}_k), \\
&\quad\vdots \\
&Y = Y_1 \oplus \bar{Y}_1, \qquad \bar{Y}_1 \supset f(\bar{Y}_2), \\
&X = Y \oplus \bar{Y}, \qquad \bar{Y} \supset f(\bar{Y}_1).
\end{aligned}$$

3. INVARIANT SUBSPACES UNDER A LINEAR MAP DEFINED ON A SUBSPACE

3.1

According to [6] [Theorem 6.6.1, with $X = \mathbb{C}^n \times \mathbb{C}^m$ and $Y = \mathbb{C}^n \equiv \mathbb{C}^n \times \{0\} \subset \mathbb{C}^n \times \mathbb{C}^m$, so that $\ker C = f^{-1}(Y)$] and [3], we define:

DEFINITION 3.1. Let $f: Y \to X$ be a linear map defined on a subspace, and $W \subset Y$ a subspace. We say that W is f-invariant if

$$f(W) \cap Y \subset W.$$

3.2. *Example*

Let $f: Y \to X$ be a linear map defined on a subspace. If $y, f(y), \dots, f^k(y)$ is a full Brunovsky chain, then the subspaces generated by subchains of the kind $f^d(y), f^{d+1}(y), \dots, f^{k-1}(y)$ are f-invariant. We will see in the next section that these subspaces are in fact "f-marked" subspaces.

3.3

Then we have the following matricial characterization:

PROPOSITION 3.1. *Let $f: Y \to X$ and $W \subset Y$ be as in Section 3.1. Then W is f-invariant if and only if there is a basis of X adapted to $W \subset Y \subset X$ such that the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ of f in this basis has the form*

$$\begin{pmatrix} A_1^1 & A_2^1 \\ 0 & A_2^2 \\ B_1 & B_2 \end{pmatrix}$$

where $A_1^1 \in M_s(\mathbb{C})$, $s = \dim W$.

Proof. Obviously, Definition 3.1 is the same as saying that there is a subspace $\bar{Y} \subset X$ such that

$$Y \oplus \bar{Y} = X, \qquad f(W) \subset W \oplus \bar{Y}.$$

Then, in any basis of X adapted to $W \subset Y \subset X$ and also to $\bar{Y} \subset X$, the matrix of f has the desired form. The converse is obvious. ∎

3.4

For a geometrical approach, we will consider the restriction of f to W,

$$\hat{f}: W \to X,$$

as a linear map defined on a subspace, and, in an analogous way to (2.3), the chain of subspaces and linear maps

$$W_0 \equiv W, \qquad W_i = \hat{f}^{-1}(W_{i-1}), \quad i > 0,$$

$$W_{h+1} = W_h \subset W_{h-1} \subset \cdots \subset W_1 \subset W \subset X,$$

$$\hat{f}_i : W_i \to W_{i-1}, \qquad i > 0.$$

Obviously, we have

$$\begin{array}{ccccccccccccc}
\cdots & \subset & Y_i & \subset & Y_{i-1} & \subset & \cdots & \subset & Y_1 & \subset & Y & \subset & X \\
 & & \cup & & \cup & & & & \cup & & \cup & & \\
\cdots & \subset & W_i & \subset & W_{i-1} & \subset & \cdots & \subset & W_1 & \subset & W & \subset & X
\end{array}$$

3.5

Then we have the following characterization:

PROPOSITION 3.2. *Let $f : Y \to X$ be a linear map defined on a subspace, and $W \subset Y$ a subspace. Then, with the above notation, the following statements are equivalent*:

(i) *W is f-invariant.*
(ii) $W_1 = W \cap Y_1$.
(iii) $W_i = W_{i-1} \cap Y_i = W \cap Y_i$ *for all* $i > 0$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is straightforward. And (iii) can be easily proved by induction:

$$\begin{aligned}
W_{i+1} &= \hat{f}^{-1}(W_i) = \hat{f}^{-1}(W \cap Y_i) = f^{-1}(W \cap Y_i) \cap W \\
&= f^{-1}(W) \cap Y_{i+1} \cap W = \hat{f}^{-1}(W) \cap Y_{i+1} = W_1 \cap Y_{i+1} \\
&= W \cap Y_i \cap Y_{i+1} = W \cap Y_{i+1}.
\end{aligned}$$

■

3.6

This geometric approach allows an alternative proof to the relation in [3] between the Brunovsky invariants of $f : Y \to X$ and those of $\hat{f} : W \to X$.

PROPOSITION 3.3 [3]. *Let $f: Y \to X$ be a linear map defined on a subspace, and $W \subset Y$ an f-invariant subspace. We denote the Brunovsky invariants of $f: Y \to X$, as in* (2.4), *and in an analogous way we denote by $(\hat{k}_1, \hat{k}_2, \dots, \hat{k}_q)$ and $\hat{\alpha}_1 \mid \hat{\alpha}_2 \mid \cdots \mid \hat{\alpha}_l$ those of $\hat{f}: W \to X$. Then, we have*

(i) *$l \leqslant t$ and $\alpha_i \mid \hat{\alpha}_i \mid \alpha_{i+t-l}$ for $i = 1, 2, \dots, l$;*
(ii) *$q \leqslant r$ and $\hat{k}_i \leqslant k_i$ for $i = 1, 2, \dots, q$.*

Conversely, given $(\hat{k}_1, \dots, \hat{k}_q)$ and $\hat{\alpha}_1 \mid \hat{\alpha}_1 \mid \cdots \mid \hat{\alpha}_l$ verifying (i) *and* (ii), *there is an f-invariant subspace $W \subset Y$ such that they are the Brunovsky invariants of $\hat{f}: W \to X$.*

Proof. Obviously $k \geqslant h$, and the subspace $W_h = W_k \subset Y_k$ is invariant under the endomorphism $f_{k+1}: Y_k \to Y_k$. Then (i) is well known [1, p. 149]. To proof (ii), we consider the conjugate partitions $(r_1, \dots, r_k)$ and $(\hat{r}_1, \dots, \hat{r}_h)$ of $(k_1, \dots, k_r)$ and $(\hat{k}_1, \dots, \hat{k}_q)$ respectively (see Section 2.4), where

$$r_i = \dim Y_{i-1} - \dim Y_i,$$

$$\hat{r}_i = \dim W_{i-1} - \dim W_i.$$

From (iii) of Proposition 3.2 we obtain $r_i \geqslant \hat{r}_i$. That obviously implies $k_i \geqslant \hat{k}_i$ and $r \geqslant q$. For the converse, it is sufficient to take as W the subspace spanned by a set of Brunovsky subchains having the desired length (see Section 3.2). ∎

3.7

As we have said in Section 3.2, we will see in the next section that the f-invariant subspace constructed in the converse of the above proof is in fact a so-called f-marked subspace. In Section 6.7, for f observable, we will determine the number of "different" subspaces of this kind for each $(\hat{k}_1, \dots, \hat{k}_q)$ verifying relation (ii).

4. MARKED SUBSPACES UNDER A LINEAR MAP DEFINED ON A SUBSPACE

4.1

In all this section, we assume $f: Y \to X$ a linear map defined in a subspace, $W \subset Y$ a f-invariant subspace, and $\hat{f}: W \to X$ the restriction of f. We maintain the notation in Sections 2 and 3.4.

4.2

Gohberg et al. [6] define the *marked* invariant subspaces under an endomorphism by means of the following condition: there is a Jordan basis of the subspace which can be extended to a Jordan basis of the global space. In an analogous way, we define:

DEFINITION 4.1. In the conditions of Section 4.1, we say that W is f-marked if there is a Brunovsky basis of $\hat{f}: W \to X$ which can be extended to a Brunovksy basis of $f: Y \to X$, or equivalently, if there is a Brunovsky basis of $f: Y \to X$ adapted to $W \subset Y \subset X$.

4.3

In a similar way to Section 3.3, the following matricial characterization is immediate:

PROPOSITION 4.1. *With the above notation, W is f-marked if and only if there is a basis of X adapted to $W \subset Y \subset X$ such that the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ of f in this basis has the form*

$$\begin{pmatrix} A_1^1 & A_2^1 \\ 0 & A_2^2 \\ B_1 & B_2 \end{pmatrix}, \qquad where \quad A_1^1 \in M_s(\mathbb{C}), \quad s = \dim W,$$

and in addition:

(i) $\begin{pmatrix} A_1^1 \\ B_1 \end{pmatrix}$ *is a Brunovsky matrix.*

(ii) $\begin{pmatrix} A \\ B \end{pmatrix}$ *is a Brunovsky matrix up to permutation; that is to say, there is a permutation matrix $P \in \mathrm{Gl}(n, \mathbb{C})$ such that*

$$\begin{pmatrix} P^t & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} P$$

is a Brunovsky matrix.

4.4. Example

Let $X = \mathbb{C}^4$, (e_1, e_2, e_3, e_4) be a basis, $Y = [e_1, e_2, e_3]$, and $f: Y \to X$ be defined by

$$f(e_1) = 0, \qquad f(e_2) = e_1 + e_3, \qquad f(e_3) = e_4.$$

Then $W = [e_3]$ is f-invariant, but it is not f-marked.

5. GEOMETRIC CHARACTERIZATION OF f-MARKED SUBSPACES

5.1

In [5] the authors characterize the f-marked subspaces for $f: X \to X$ an endomorphism. In fact, for f nilpotent, they prove that an invariant subspace $W \subset X$ is marked if and only if

$$W \cap f^{d+1}(K_{d+h+1}) + W \cap f^{d}(K_{d+h-1})$$

$$= W \cap \left[f^{d+1}(K_{d+h+1}) + f^{d}(K_{d+h-1})\right]$$

for all d, h, where $K_l = \ker f^l = f^{-l}(0)$.

Here, for $f: Y \to X$ a linear map defined on a subspace, we obtain a similar geometric characterization of the f-marked subspaces $W \subset Y$ in terms of the subspaces $Y_i = f^{-1}(Y)$ (see Section 2.3) and $W_i = \hat{f}^{-1}(W)$ (see Section 3.4). It is proved in Sections 5.3, 5.4, and 5.5.

THEOREM 5.1. *Let $f: Y \to X$ be a linear map defined on a subspace, and $W \subset Y$ an f-invariant subspace. Then W is f-marked if and only if*

(i) *W_k is f_{k+1}-marked,*
(ii) $[W \cap f^{j-i}(Y_{j-1})] + Y_i = (W + Y_i) \cap [f^{j-i}(Y_{j-1}) + Y_i]$

for all $1 \leqslant i \leqslant j \leqslant k$.

5.2. Remarks

REMARK 1. Because $f^{j-i}(Y_{j-1}) \subset Y_{i-1}$ and $W \cap Y_{i-1} = W_{i-1}$ (see Section 3.5), we can replace W by W_{i-1} in both members of (ii). Furthermore, the right inclusion in (ii) is always true.

Hence, condition (ii) above is equivalent to

(ii′) $[W_{i-1} \cap f^{j-i}(Y_{j-1})] + Y_i \supset (W_{i-1} + Y_i) \cap [f^{j-i}(Y_{j-1}) + Y_i]$.

REMARK 2. Notice that this inclusion (ii′) is always true for $i = j$, and for $i > h$.

REMARK 3. X. Puerta has remarked that condition (ii) is equivalent to

(ii″) $W \cap [f^{j-i}(Y_{j-1}) + Y_i] = W \cap f^{j-i}(Y_{j-1}) + W \cap Y_i$,

which is much more similar to the one for f a nilpotent endomorphism.

5.3. *Proof of the Necessity*
(i) is obvious.
(ii): With the notation in Section 2.5, we have, for all $1 \leqslant i \leqslant j \leqslant k$,

$$f^{j-i}(Y_{j-1}) + Y_i = \left[f^{j-i}(Y_j) \oplus f^{j-i}(\bar{Y}_j)\right] + Y_i = f^{j-i}(\bar{Y}_j) + Y_i$$

(in fact, the last sum is a direct sum). The Brunovsky bases of $\hat{f}: W \to X$ are obtained in an analogous way to (2.5). In particular, for all $1 \leqslant i \leqslant k$,

$$W_{i-1} + Y_i = \overline{W}_i + Y_i$$

(here also, the last sum is a direct sum).

Since W is f-marked, there exist Brunovsky bases of $f: Y \to X$ and $\hat{f}: W \to X$ such that

$$\overline{W}_i = \bar{Y}_i \cap W \qquad (1 \leqslant i \leqslant k).$$

Hence, for all $1 \leqslant i < j \leqslant k$,

$$\left[\overline{W}_i + f^{j-i}(\bar{Y}_j)\right] \cap Y_i = 0.$$

Therefore

$$(\overline{W}_i + Y_i) \cap \left[f^{j-i}(\bar{Y}_j) + Y_i\right] = \left[\overline{W}_i \cap f^{j-i}(\bar{Y}_j)\right] + Y_i,$$

and inclusion (ii′) in Section 5.2 follows immediately.

5.4
To prove the sufficiency, we will need the following lemma:

LEMMA 5.1. *In the conditions of Section* 4.1, *let*

$$W_j^i = \frac{(W_{i-1} + Y_i) \cap \left[f^{j-i}(Y_{j-1}) + Y_i\right]}{(W_{i-1} + Y_i) \cap \left[f^{j-i+1}(Y_j) + Y_i\right]}, \qquad 1 \leqslant i \leqslant j \leqslant k.$$

Then

(a) *We have natural isomorphisms*

$$\frac{W}{W_k} \cong \bigoplus_{i=1}^{k} \frac{W_{i-1}}{W_i},$$

$$\frac{W_{i-1}}{W_i} \cong \bigoplus_{j=i}^{k} W_j^i \qquad (1 \leqslant i \leqslant k).$$

(b) *For all* $1 \leqslant i \leqslant j \leqslant k$, *the linear maps*

$$\tilde{f} : W_j^i \to W_j^{i-1}$$

induced by f *are injective.*

Proof. (a): The first isomorphism is obvious, bearing in mind the stationary chain

$$\cdots = W_k = \cdots = W_h \subset W_{h-1} \subset \cdots \subset W_1 \subset W.$$

For the second one, we remark that, because W is f-invariant, we have (see Section 3.7) $W_i = W_{i-1} \cap Y_i$, so that

$$\frac{W_{i-1}}{W_i} = \frac{W_{i-1}}{W_{i-1} \cap Y_i} \cong \frac{W_{i-1} + Y_i}{Y_i}.$$

Now, we consider the finite chain

$$Y_i = f^{k+1-i}(Y_k) + Y_i \subset \cdots \subset f^{j-i}(Y_{j-1}) + Y_i \subset \cdots \subset Y_{i-1} + Y_i = Y_{i-1}.$$

If we intersect each term with $W_{i-1} + Y_i$, we have

$$Y_i \subset \cdots \subset (W_{i-1} + Y_i) \cap \left[f^{j-i}(Y_{j-1}) + Y_i \right] \subset \cdots \subset W_{i-1} + Y_i,$$

and the desired isomorphism follows immediately.

(b): Let $x \in f^{j-i}(Y_{j-1}) + Y_i$, with $\tilde{f}(\tilde{x}) = 0$. Then there exists $y \in Y_j$ such that $f(x) - f^{j-i+2}(y) \in Y_{i-1}$. Hence $x - f^{j-i+1}(y) \in Y_i$, and $\tilde{x} = 0$. ∎

5.5. Proof of the Sufficiency

Firstly, because of condition (i) we can obtain a Jordan basis of Y_k adapted to W_k. Hence it is sufficient to construct a Brunovsky basis of a complementary subspace of W_k to W which can be extended to a Brunovsky basis of a complementary subspace of Y_k to Y. In order to do so, we can sketch the above lemma in the following diagram

$$\begin{array}{lllllll}
W/W_k & & & & & & \\
\cong & & & & & & \\
W_{k-1}/W_k & \cong & W_k^k & & & & \\
\oplus & & & \searrow & & & \\
W_{k-2}/W_{k-1} & \cong & W_{k-1}^{k-1} & \oplus & W_k^{k-1} & & \\
\oplus & & & \searrow & & \searrow & \\
W_{k-3}/W_{k-2} & \cong & W_{k-2}^{k-2} & \oplus & W_{k-1}^{k-2} & \oplus & W_k^{k-2} \\
\oplus & & & & & & \\
\vdots & & & & & & \\
\oplus & & & & & & \\
W/W_1 & \cong & W_1^1 & \oplus & W_2^1 & \oplus\ \cdots & \oplus W_k^1
\end{array}$$

But now, if condition (ii) in Section 5.1 (or equivalently, condition (ii′) in Section 5.2) is verified, we have, for all $1 \leqslant i \leqslant j \leqslant k$

$$W_j^i = \frac{\left[W_{i-1} \cap f^{j-i}(Y_{j-1})\right] + Y_i}{\left[W_{i-1} \cap f^{j-i+1}(Y_j)\right] + Y_i} \cong \frac{W_{i-1} \cap f^{j-i}(Y_{j-1})}{W_{i-1} \cap f^{j-i+1}(Y_j) + W_i \cap f^{j-i}(Y_{j-1})}$$

(the last step is a direct application of the general natural isomorphism

$$\frac{F + G}{F' + G} \cong \frac{F}{F' + F \cap G},$$

where $F' \subset F$).

Thus, we can obtain an extendible Brunovsky basis of W by means of taking images and extending step by step (see Section 2.5), as we sketch in the following diagram:

$$\begin{array}{lllll}
\hat{B}_k^k & & & & \\
\hat{B}_{k-1}^{k-1} & f(\hat{B}_k^k), \hat{B}_k^{k-1} & & & \\
\hat{B}_{k-2}^{k-2} & f(\hat{B}_{k-1}^{k-1}), \hat{B}_{k-1}^{k-2} & f^2(\hat{B}_k^k), f(\hat{B}_k^{k-1}), \hat{B}_k^{k-2} & & \\
\vdots & \vdots & \ddots & \ddots & \\
\hat{B}_1^1 & f(\hat{B}_2^2), \hat{B}_1^1 & \cdots & \cdots & f^{k-1}(\hat{B}_k^k), \ldots, f(\hat{B}_k^2), \hat{B}_k^1
\end{array}$$

That is to say, we take:

for $j = k, k-1, \dots, 1$: $\hat{B}_j^j \subset W_{j-1} \cap Y_{j-1}$ such that its classes form a basis of W_j^j,

for $j = k, k-1, \dots, 2$: $\hat{B}_j^{j-1} \subset W_{j-2} \cap f(Y_{j-1})$ such that the classes of $f(\hat{B}_j^j)$, $\hat{B}_j^{j-1}$ form a basis of W_j^{j-1},

...

for $j = k$: $\hat{B}_j^{j-k+1} \subset W_{j-k} \cap f^{k-1}(Y_{j-1})$ such that the classes of $f^{k-1}(\hat{B}_j^j), \dots, f(\hat{B}_j^{j-k+2}), \hat{B}_j^{j-k+1}$ form a basis of W_j^{j-k+1}.

Obviously, the above sets $\hat{B}_j^i$, $1 \leqslant i \leqslant j \leqslant k$, and their images form a Brunovsky basis $\hat{B}$ of the complementary subspace $\oplus_{i=1}^k \overline{W}_i$ of W_k to W where

$$\overline{W}_i = \left[\bigcup_{\substack{i \leqslant j \leqslant k \\ 0 \leqslant l \leqslant j-i}} f^l\left(\hat{B}_j^{i+l}\right) \right], \qquad 1 \leqslant i \leqslant k.$$

And this basis is extendible to a Brunovsky basis of a complementary subspace of Y_k to Y because

$$\hat{B}_j^i \subset W_{i-1} \cap f^{j-i}(Y_{j-1}), \qquad 1 \leqslant i \leqslant j \leqslant k.$$

In fact, first we extend the chains in $\hat{B}$ to their maximal length, taking antiimages

$$f^{-1}\left(\hat{B}_j^i\right), \dots, f^{-(j-i)}\left(\hat{B}_j^i\right), \qquad 1 \leqslant i < j \leqslant k,$$

where this doesn't mean the set of antiimages of $\hat{B}_j^i$, but a chain of antiimages of each vector in $\hat{B}_j^i$.

Finally, if we write

$$\hat{Y}_i = \bigoplus_{j \geqslant i}^{k} \left[f^{j-i}\left(\hat{B}_j^j\right), f^{j-i-1}\left(\hat{B}_j^{j-1}\right), \dots, \hat{B}_j^i, f^{-1}\left(\hat{B}_j^{i-1}\right), \dots, f^{-i+1}\left(\hat{B}_j^1\right) \right]$$

for $1 \leqslant i \leqslant k$, we have

$$Y_i \oplus \overline{W}_i \subset Y_i \oplus \hat{Y}_i \subset Y_{i-1},$$

$$f\left(\hat{Y}_i\right) \subset \hat{Y}_{i-1}.$$

Therefore, we can construct $\overline{Y}_k, \overline{Y}_{k-1}, \dots$ (see Section 2.5), such that $\overline{Y}_i \supset \hat{Y}_i$. Thus, we have extended the basis $\overline{B}$ of $\oplus_{k=1}^{k} \overline{W}_i$ to a Brunovsky basis of the complementary subspace $\oplus_{i=1}^{k} \overline{Y}_i$ of Y_k to Y, as we desired.

6. CLASSIFICATION OF f-MARKED SUBSPACES

6.1

We consider the following natural equivalence relation between two f-marked subspaces:

DEFINITION 6.1. Let $f: Y \to X$, $f': Y' \to X'$ be two linear maps defined on a subspace, and $W \subset Y$, $W' \subset Y'$ marked subspaces under them respectively. We say that they are equivalent if there is an isomorphism $\varphi: X \to X'$ such that

$$\varphi(Y) = Y', \qquad \varphi(W) = W',$$

$$\varphi \circ f = f' \circ \hat{\varphi},$$

where $\hat{\varphi}$ is the restriction of φ to Y.

6.2

In particular, f and $\hat{f}$ must be block-similar to f' and $\hat{f'}$ respectively. Obviously, we must suppose $\dim X = \dim X'$, $\dim Y = \dim Y'$, $\dim W = \dim W'$. From now on, these dimensions will be respectively $n + m$, n, and s.

6.3

The construction in Sections 5.4 and 5.5 suggests considering the invariants $\dim W_j^{\,i}$:

DEFINITION 6.2. Let $f: Y \to X$ be a linear map defined on a subspace, and $W \subset Y$ an f-marked subspace. We consider the numerical invariants

$$w_j^i = \dim(W + Y_i) \cap \left[f^{j-i}(Y_{j-1}) + Y_i\right]$$

$$- \dim(W + Y_i) \cap \left[f^{j-i+1}(Y_j) + Y_i\right]$$

for $1 \leqslant i \leqslant j \leqslant k$. For convenience, we define $w_j^i = 0$ if $1 \leqslant j < i$.

6.4

From (a) in Lemma 5.1 it is obvious that $\hat{r}_i = \dim W_{i-1} - \dim W_i = w_i^i + \cdots + w_k^i$ $(1 \leqslant i \leqslant k)$, and we recall that the conjugate partition of $(\hat{r}_1, \dots, \hat{r}_k)$ gives the Kronecker indices of $\hat{f}: W \to X$.

6.5

Then we have the following classification result.

THEOREM 6.1. *Let $f: Y \to X$, $f': Y' \to X'$ be two linear maps defined on a subspace, and $W \subset Y$, $W' \subset Y'$ marked subspaces under them respectively. Then they are equivalent if and only if:*

(i) *f and f' are block-similar.*

(ii) *W_k and W_k' are equivalent as marked subspaces under the endomorphisms f_{k+1} and f_{k+1}', respectively.*

(iii) *$w_j^i = w_j'^i$ for all $1 \leqslant i \leqslant j \leqslant k$.*

Proof. The three conditions are obviously necessary. For the sufficiency, we will consider the construction of extendible Brunovsky bases in Section 5.5, to show that a bijection between Brunovsky bases of X and X', adapted to $W \subset Y \subset X$ and $W' \subset Y' \subset X'$ respectively, and commuting with f and f', is possible. Because of condition (ii), this is trivial for the Jordan bases of Y_k and Y_k'. Moreover, condition (iii) implies that the numbers of vectors in the sets $\hat{B}_j^i$, which generate the Brunovsky chains for $\hat{f}$, are the same as the numbers of the analogous ones for $\hat{f}'$; then, if we take bijections $\hat{B}_j^i \to \hat{B}_j'^i$, $1 \leqslant i \leqslant j \leqslant k$, and we extend these bijections naturally to the corresponding generated chains, we obtain a bijection $\hat{B} \to \hat{B}'$ which commutes with f and f'. Now, we extend the bijection to the antiimages $f^{-l}(\hat{B}_j^i)$, so that we have isomorphisms $\hat{Y}_i \to \hat{Y}_i'$, $1 \leqslant i \leqslant k$, always commuting with f and f'. Finally, condition (i) ensures that $\bar{Y}_i, \bar{Y}_i'$ can be chosen so that the above isomorphisms can be extended to $\bar{Y}_i \to \bar{Y}_i'$, $1 \leqslant i \leqslant k$, commuting with f and f'. ■

6.6

We remark that [5] gives conditions for (ii) above to be verified.

6.7

As an application, we can determine how many nonequivalent marked subspaces verify the converse in Proposition 3.3 for f observable.

Proposition 6.1. *Let $f: Y \to X$ be an observable linear map defined on a subspace, with Kronecker indices $k_1 \geqslant \cdots \geqslant k_r$. Then there are as many (up to equivalence) f-marked subspaces $W \subset Y$ as collections of integers $(m_1, \dots, m_r)$ verifying the following conditions*:

(i) $k_i \geqslant m_i \geqslant 0$;
(ii) $m_i \geqslant m_{i+1}$ *for the indices* $1 \leqslant i \leqslant r$ *such that* $k_i = k_{i+1}$.

In particular, given $\hat{k}_1 \geqslant \cdots \geqslant \hat{k}_r$ with $\hat{k}_i \leqslant k_i$ $(1 \leqslant i \leqslant r)$, there are as many (up to equivalence) f-marked subspaces $W \subset Y$ with Kronecker indices $(\hat{k}_1, \dots, \hat{k}_r)$ as permutations $(m_1, \dots, m_r)$ of them verifying (i) *and* (ii).

Proof. In Theorem 6.1 we saw that an f-marked subspace is characterized up to equivalence by the numbers w_j^i, $1 \leqslant i \leqslant j \leqslant k$. These numbers must verify

(a) $w_j^i \geqslant w_j^{i+1} \geqslant 0$,
(b) $w_j^1 \leqslant r_j - r_{j+1}$,

and conversely, it is obvious that for every family of numbers w_j^i, $1 \leqslant i \leqslant j \leqslant k$, verifying (a) and (b) there exists an f-marked subspace which has these invariants. Therefore, we have proved that there is a bijection between the sets of numbers $\{w_j^i\}$ verifying conditions (a)–(b) and the equivalence classes of f-marked subspaces.

To conclude, we will prove that there exists a bijection between the collections of numbers $\{w_j^i\}$ verifying (a)–(b) and those of numbers $\{m_i\}$ verifying conditions (i)–(ii) in the proposition.

Given $\{w_j^i\}_{1 \leqslant i \leqslant j \leqslant k}$ with conditions (a)–(b), if $r_{j+1} < i \leqslant r_j$, we define $m_i = \operatorname{card}_{1 \leqslant l \leqslant j}\{w_j^l : w_j^l \geqslant i - r_{j+1}\}$, and if $w_j^1 > 0$, then $m_{r_{j+1}+1} \geqslant \cdots \geqslant m_{r_j}$ is the conjugated partition of $w_j^1 \geqslant \cdots \geqslant w_j^j$ filled by zeros if $w_j^1 < r_j - r_{j+1}$.

Let us see that this collection verifies conditions (i)–(ii) in the proposition.

(i): If $r_{j+1} < i \leqslant r_j$, then obviously $m_i \leqslant j$. But, because of the relation between k-numbers and r-numbers, we have $k_{r_{j+1}+1} = \cdots = k_{r_j} = j$.

(ii): If $k_i = k_{i+1}$, there exists j $(1 \leqslant j \leqslant k)$ such that $r_{j+1} < i < i + 1 \leqslant r_j$, and then $m_i \geqslant m_{i+1}$.

Conversely, let be a collection of m_i verifying (i)–(ii). Then we define $w_j^1 \geqslant \cdots \geqslant w_j^j$ as the conjugate partition of $m_{r_{j+1}+1} \geqslant \cdots \geqslant m_{r_j}$ if $r_{j+1} < r_j$, and conditions (a) and (b) are obvious. ∎

Finally, we remark that the numbers $\{m_i\}$ are the lengths of the Brunovsky chains of $\hat{f}: W \to X$. Hence, they are their Kronecker indices $\{\hat{k}_i\}$ up to permutation.

REFERENCES

1 N. Bourbaki, *Eléments de Mathématique*, Algèbre II, Hermann, Paris, 1964.
2 R. Bru, L. Rodman, and H. Schneider, Extensions of Jordan bases for invariant subspaces of a matrix, *Linear Algebra Appl.* 150:209–225 (1991).
3 I. Baragaña and I. Zaballa, Block similarity invariants of restrictions to (A, B)-invariant subspaces, *Linear Algebra Appl.* 220:31–62 (1995).
4 J. Ferrer and F. Puerta, Similarity of non-everywhere-defined linear maps, *Linear Algebra Appl.* 150:27–55 (1992).
5 J. Ferrer, F. Puerta, and X. Puerta, Geometric characterization and classification of marked subspaces, *Linear Algebra Appl.*, 235:15–34 (1996).
6 I. Gohberg, P. Lancaster, and L. Rodman, *Invariant Subspaces of Matrices with Applications*, Wiley, New York, 1986.

Received 15 February 1995; final manuscript accepted 6 November 1995